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Conservation laws of discrete Lax equations

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Abstract

Based on the pseudodifference operators' ring and Lax pair, we present a residue formula to calculate the conservation laws of discrete Lax equations. The details are explained with two illustrative examples. Furthermore, some numerical results are given to verify the method.

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1. Introduction

Conservation laws are important in both mathematics and physics. Although the famous Noether's theorem has achieved great success in seeking the conservation laws of many systems (see, e.g., [7]), it is still a nontrivial task to obtain the conservation laws for a given system. Breakthroughs took place in the late 1960s when Miura *et al* found that the KdV equation possesses infinitely many conservation laws. From then on, conservation laws became one of the central ideas for integrable systems and some techniques were established to derive them. Besides, there were also some works aimed directly at conservation laws. Zhang *et al* [11] obtained a certain class of conserved quantities for some integrable systems by mastersymmetries and recursion operators. Zhang [10] analysed the q-deformation of KdV hierarchies and expressed the conservation laws by integration. Hereman *et al* presented some powerful direct methods [4] for general differential or difference systems to compute the first few conserved densities and fluxes.

In this paper, we will consider the conservation laws of integrable differential-difference equations (IDDEs) of Lax type. We derive the conservation laws by calculating the residues of related pseudodifference operators (PDOs). It is the discrete analogue of the differential case, so it will explicitly provide many conservation laws for a discrete system. The key step in the calculations is a residue formula named by us. Due to the discrete nature of PDOs, the calculations of the residues are usually nontrivial tasks. To help the calculations, a special technique is introduced. We choose two illustrative examples to demonstrate the method. The surprising point is that the number of the conservation laws provided by the residue formula is more than the number of the eigenvalues of the Lax operator (at least in some cases of the first example).

The periodic boundary condition is one of the most interesting cases in computation and application. Therefore, we will restrict our arguments to the periodic boundary condition.

In section 2, we first present the theory of PDOs. Then by analysing a series of commutators we establish a general discrete Lax equation. At the end of this section, we give the residue formula for calculating the conserved quantities of the discrete Lax equation. In section 3, we present two examples of discrete Lax systems and calculate their conserved quantities. In section 4, we summarize the conclusions.

2. Theory of PDOs and construction of discrete Lax equations

In this section, we present the theory of PDOs and the method of constructing discrete Lax equations by PDOs.

2.1. Constraints imposed by the boundary condition and symmetry

One way to generate an integrable system is by the Lax equation (see, e.g., [3])

$$\dot{L} = [P, L], \tag{1}$$

where P and L are both linear operators. For example, by setting

$$L = \partial^2 + u, \qquad P = \partial^3 + \frac{3}{2}u\partial + \frac{3}{4}u', \qquad \partial = \frac{\mathrm{d}}{\mathrm{d}x},$$

we get KdV equation

$$4\dot{u} = u''' + 6uu',$$

where $u = u(x, t)$, $\dot{u} = \frac{\partial}{\partial t}u$ and $u' = \frac{\partial}{\partial x}u$. The systematical analysis of Lax equation (1) involves the ring of pseudodifferential operators [3]. In the continuous case, the ring of pseudodifferential operators and the boundary condition are relatively independent. That is to say, the boundary condition will not affect the definitions of the operators in the pseudodifferential operators' ring. For example, operator ∂ is always defined as $\frac{\mathrm{d}}{\mathrm{d}x}$ no matter what boundary condition we take. Therefore, for any function f the Leibniz rule $\partial f = f\partial + f'$ will not be affected by the boundary condition. So do the other formulae. The situation changes in the discrete case. The boundary condition will impose some constraints on the formal operators.

Let us first consider the algebraization of the shift operator Λ (see, e.g., [1, 8]). The matrix representation of the operator Λ is

$$\Lambda_{ij} = \delta_{i,j-1}.$$

The Lie algebra

$$\mathcal{D} = \left\{ \sum_{-\infty<i\ll\infty} a_i\Lambda^i, \quad a_i \text{ diagonal operators} \right\} = \mathcal{D}_+ + \mathcal{D}_-,$$

where

$$\mathcal{D}_+ = \left\{ \sum_{0\leqslant i\ll\infty} a_i\Lambda^i \in \mathcal{D} \right\}, \qquad \mathcal{D}_- = \left\{ \sum_{-\infty<i<0} a_i\Lambda^i \in \mathcal{D} \right\},$$

is the foundation of the theory. Denote the number of grid points by N. We immediately get $\Lambda^N = \mathbf{1}$ by the periodic boundary condition. Therefore, the inverse of Λ is Λ^{N-1}, i.e., $\Lambda^{-1} = \Lambda^{N-1}$. Now it is clear that we cannot split $\mathcal{D}$ into $\mathcal{D}_+$ and D_-. Hence, the algebraization of the shift operator does not meet our demands.

We may also consider the q-pseudodifference operator. However, the adoption of q-pseudodifference operator will cause at least two difficulties. One difficulty is that it is hard to cope with the periodic boundary condition. In fact, if we adopt the q-pseudodifference operator and the periodic boundary condition at the same time, then there is not a reasonable limit for the system. The other difficulty is that it is hard to deal with a translation invariant system, because the discrete equations obtained by the q-pseudodifference operators cannot manifestly keep the symmetry of translation.

In the following sections, it will be clear that the algebraization of the operator Δ [5, 6] suffices for all the constraints. Another hint for adopting pseudodifference is that Haine *et al* [5] revealed a close relationship between the differential and the ordinary difference.

2.2. *Summary of PDOs ring: definition and basic formula*

Here, we will present the basic ideas about PDOs. These materials are mainly borrowed from [5]. There some other useful material can be found.

To understand the context of PDOs, it is convenient to compare it with the well-known pseudodifferential operators. The principal theories of PDOs are parallel to that of pseudodifferential ones. Therefore, we will first recall the basic ideas about pseudodifferential operators.

The first idea in the theory of pseudodifferential operators is to view the derivative ∂ as a linear operator acting on function space $\mathcal{F} = \{f \,|\, f(x) \in \mathbb{C}\}$:

$$(\partial f)(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x).$$

In addition, any function g is a diagonal operator acting on $\mathcal{F}$:

$$(g \circ f)(x) = g(x) f(x).$$

So, we should consider the minimal ring of operators containing derivative operator ∂ and all diagonal operators f. In this differential operators' ring, formula

$$\partial f = f_x \partial + f \partial \tag{2}$$

is of basic importance. In fact, formula (2) is just the Leibniz rule

$$\begin{aligned} \partial \circ f \circ g &= \partial(fg) \\ &= (f_x) g + f \partial g \\ &= f_x \circ g + f \circ \partial \circ g, \end{aligned} \tag{3}$$

where f, f_x are operators and g is an arbitrary function in $\mathcal{F}$. The direct result of (2) is the interim operator formula

$$\partial^n f = \sum_{k=0}^{\infty} \binom{n}{k} f^{(k)} \partial^{n-k}, \qquad n \in Z_+. \tag{4}$$

The second idea in the theory of pseudodifferential operators is to replace Z_+ in (4) by Z. Thus, we have

$$\partial^n f = \sum_{k=0}^{\infty} \binom{n}{k} f^{(k)} \partial^{n-k}, \qquad n \in Z. \tag{5}$$

For example, $\partial^{-1} f = f \partial^{-1} - f_x \partial^{-2} + f_{xx} \partial^{-3} - \cdots$. Finally, formula (5) defines the pseudodifferential operators' ring. All of the above is well known.

In the discrete case, the situation is a little more complicated. Let h be a real constant. The forward difference and backward difference

$$(\Delta_f \cdot g)(i) = \frac{g(i+1) - g(i)}{h},$$
$$(\Delta_b \cdot g)(i) = \frac{g(i) - g(i-1)}{h},$$

where $g(i)$ denotes the ith component of vector g, are the two simplest choices. For the convenience of the following paper, here we use the symbol $\Delta_f \cdot g$ but not $\Delta_f g$ to denote the forward difference of vector g. For simplicity, we only consider the forward difference. The theory for the backward difference is similar. So, we define

$$\Delta = \Delta_f.$$

Difference Δ is an operator acting on a vector space V of arbitrary but definite dimension N. Parallel to the continuous case, any vector v of dimension N can be identified as an operator by

$$(v \circ f)(i) = v(i) f(i), \qquad i = 1, 2, \ldots, N.$$

So we will consider the minimal ring containing Δ and all vectors of dimension N. The basic formula for the ring is derived by rewriting the modified Leibniz rule

$$\Delta \cdot (fg) = (\Delta \cdot f)g + (f + h\Delta \cdot f)\Delta \cdot g \tag{6}$$

into an operator form

$$\Delta \circ f \circ g = (\Delta \cdot f) \circ g + (f + h\Delta \cdot f) \circ \Delta \circ g. \tag{7}$$

In (6), the multiplication among vectors f, $\Delta \cdot f$ and g is component multiplication, i.e., $(fg)(i) = f(i)g(i)$. In (7), f and $\Delta \cdot f$ are operators acting on V and g is a vector in V. But in (7) vector g is arbitrary, so we have the operator equation

$$\Delta \circ f = (\Delta \cdot f) + (f + h\Delta \cdot f) \circ \Delta. \tag{8}$$

For brevity, from now on we will omit the composition symbol $\circ$ in operator expressions. Based on (8), we can immediately derive the interim

$$\Delta^n f = \sum_{k=0}^{\infty} \binom{n}{k} (\Delta^k (1 + h\Delta)^{n-k}) \cdot f \Delta^{n-k}, \qquad n \in Z_+. \tag{9}$$

Note that the definition of $\Delta^n \cdot f$ is $\Delta \cdot (\Delta^{n-1} \cdot f)$. Formula (8) is the basic formula for the difference operators' ring. Finally, as continuous case, we replace Z_+ in (9) by Z to get the basic formula

$$\Delta^n f = \sum_{k=0}^{\infty} \binom{n}{k} (\Delta^k (1 + h\Delta)^{n-k}) \cdot f \Delta^{n-k}, \qquad n \in Z \tag{10}$$

for the PDOs ring.

The ring of PDOs itself is first a vector space

$$V_\Delta = \left\{ \sum_{i=-\infty}^{\infty} f_i \Delta^i, \quad f_i \text{ are operators corresponding to vectors of dimension } N \right\} \tag{11}$$

with dimension infinity. Furthermore, we can prove that the multiplication (10) is associative. Hence, V_Δ is a ring.

Let $v = \sum_{i=-\infty}^{m} f_i \Delta^i$, $f_m \neq 0$. The order of v is defined as m. We denote it by Order(v), i.e.,

$$\text{Order}(v) = m.$$

For the order of elements in V_Δ, we have the following result.

Theorem 1. *If $\hat{A} \in V_\Delta$ and $\hat{B} \in V_\Delta$, then*

$$\text{Order}(\hat{A}\hat{B}) = \text{Order}(\hat{A}) + \text{Order}(\hat{B}).$$

Let $\hat{A} = \sum_{i=-\infty}^{n} a_i \Delta^i$, $a_n \neq 0$. We introduce the following definitions.

Definition 1.

$$\text{Res}(\hat{A}) = a_{-1}.$$

Definition 2.

$$\text{Pos}(\hat{A}) = \begin{cases} \sum_{i=0}^{n} a_i \Delta^i, & \text{if} \quad n \geqslant 0, \\ 0, & \text{if} \quad n < 0. \end{cases}$$

Definition 3.

$$\hat{R} = 1 + h\Delta.$$

We also call Res($\hat{A}$) the residue of $\hat{A}$.

2.3. A series of commutators

The commutator of $\hat{A}$ and $\hat{B}$ is not necessarily $\hat{A}\hat{B} - \hat{B}\hat{A}$. For example, in [9] the q-commutator $[A, B]_q = (DA)B - BA$ has been defined. In fact, the choice of commutator is important and subtle for the theory of discrete Lax system. It is a nontrivial task to find meaningful commutators. Among the meaningful commutators, the ordinary commutator $[\cdot, \cdot]$ is proved to be one of them. In the following, we will see the commutator $[\cdot, \cdot]_1$

$$[\hat{A}, \hat{B}]_1 = \hat{A}\hat{B} - \hat{B}(\hat{R} \cdot \hat{A}) \tag{12}$$

is also useful.

First, we define

$$\hat{A}_+ = \hat{R}\hat{A}\hat{R}^{-1}, \tag{13}$$

$$\hat{A}_- = \hat{R}^{-1}\hat{A}\hat{R}. \tag{14}$$

One equivalent statement of (13) and (14) is

$$\hat{A}_+\hat{R} = \hat{R}\hat{A}, \tag{15}$$

$$\hat{R}\hat{A}_- = \hat{A}\hat{R}. \tag{16}$$

We also define

$$\hat{A}_{n+} = \hat{R}^n\hat{A}\hat{R}^{-n}, \tag{17}$$

$$\hat{A}_{n-} = \hat{R}^{-n}\hat{A}\hat{R}^n. \tag{18}$$

Let $\hat{A} = \sum_{j=-\infty}^{n} a_j \Delta^j$. With (15), we can easily verify

$$\hat{A}_+ = \sum_{j=-\infty}^{n} \hat{R} \cdot a_j \Delta^j = \hat{R} \cdot \hat{A}.$$

In fact for any PDO $\hat{A}$, we can easily derive the following formulae:

$$\hat{A}_{k+} = \hat{R}^k \cdot \hat{A}, \qquad \hat{A}_{k-} = \hat{R}^{-k} \cdot \hat{A}.$$

The following lemma explains why we introduce the commutator $[\cdot, \cdot]_1$.

Lemma 1. *For any PDO $\hat{A}$ and $\hat{B}$, there exists an N-dimensional vector H, such that*

$$\mathrm{Res}([\hat{A}, \hat{B}]_1) = \Delta \cdot H,$$

here N still denotes the number of grid points.

For example, let a and b are vectors, $\hat{A} = a\Delta, \hat{B} = b\Delta^{-1}$. Then $[\hat{A}, \hat{B}]_1 = a\Delta b\Delta^{-1} - b\Delta^{-1}a_+\Delta = (ab_+ - ba) + \Delta \cdot (ba_-)\Delta^{-1} + \cdots$. Hence $H = ba_-$. Note that in the previous example $[\hat{A}, \hat{B}] = a\Delta b\Delta^{-1} - b\Delta^{-1}a\Delta = (ab_+ - ba_-) + (a\Delta \cdot b + b\Delta \cdot a_{2-})\Delta^{-1} + \cdots$. Hence $\mathrm{Res}([\hat{A}, \hat{B}]) = a\Delta \cdot b + b\Delta \cdot a_{2-}$. So the residue of $[\hat{A}, \hat{B}]$ is not necessarily of the form $\Delta \cdot H$.

One immediate consequence of lemma 1 is

$$\sum_{i=0}^{N-1} \hat{R}^i \cdot \mathrm{Res}([\hat{A}, \hat{B}]_1) = 0. \tag{19}$$

We have taken advantage of the periodic boundary condition to get (19).

Now for integer m, we define the series of commutators as

$$[\hat{A}, \hat{B}]_m = \hat{A}\hat{B} - \hat{B}\hat{A}_{m+}. \tag{20}$$

Obviously, the commutators $[\cdot, \cdot]$ and $[\cdot, \cdot]_1$ are two special cases of (20).

We can directly verify the following results.

Corollary 1.

$$[\hat{A}, \hat{B}]_m \hat{R}^{m-1} = \hat{A}(\hat{B}\hat{R}^{m-1}) - (\hat{B}\hat{R}^{m-1})\hat{A}_+, \tag{21}$$

$$[\hat{A}, \hat{B}]_m \hat{R}^m = \hat{A}(\hat{B}\hat{R}^m) - (\hat{B}\hat{R}^m)\hat{A}. \tag{22}$$

2.4. Get discrete Lax equations

We introduce the discrete Lax equation as

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{L} = [\hat{P}, \hat{L}]_m, \tag{23}$$

where $m \in Z$. According to (19) and (21), the conserved quantity of (23) is

$$j_1 = \sum_{i=0}^{N-1} \hat{R}^i \cdot \mathrm{Res}(\hat{L}\hat{R}^{m-1}), \tag{24}$$

i.e., $\frac{\mathrm{d}}{\mathrm{d}t} j_1 = 0$.

Suppose

$$\underbrace{\hat{X}\hat{R}^m\hat{X}\hat{R}^m \cdots \hat{R}^m\hat{X}}_{n\hat{X}} = \hat{L}. \tag{25}$$

Then, we can conclude

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{X} = [\hat{P}, \hat{X}]_m. \tag{26}$$

We call $\hat{X}$ the nth root of $\hat{L}$. Let

$$\hat{Y}_k = \underbrace{\hat{X}\hat{R}^m\hat{X}\hat{R}^m \cdots \hat{R}^m\hat{X}}_{k\hat{X}}. \tag{27}$$

Then, we can verify

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{Y}_k = [\hat{P}, \hat{Y}_k]_m. \tag{28}$$

According to (19) and (21),

$$j_{\frac{k}{n}} = \sum_{i=0}^{N-1} \hat{R}^i \cdot \mathrm{Res}(\hat{Y}_k\hat{R}^{m-1}), \qquad k = 1, 2, \ldots, \tag{29}$$

are all constants of motion. We call formula (29) *residue formula*, which is the base for our calculations of the conserved quantities of the discrete Lax system.

Note that the operator $\hat{P}$ in (23) is not arbitrary. It depends on the operator $\hat{L}$. For $\hat{L} = \Delta^n + \sum_{i=-\infty}^{n-1} l_i \Delta^i$, we can choose $\hat{P}$ as

$$\hat{P}_s = \mathrm{Pos}((\hat{X}\hat{R}^m)^s), \qquad s \in N, \tag{30}$$

where $\hat{X}$ is the nth root of $\hat{L}$.

The following theorem states the equivalence of (23) for different m.

Theorem 2.

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{L} = [\hat{P}, \hat{L}]_m \rightleftharpoons \frac{\mathrm{d}}{\mathrm{d}t}(\hat{L}\hat{R}^m) = [\hat{P}, \hat{L}\hat{R}^m]. \tag{31}$$

However, theorem 2 does not state that the commutator $[\cdot, \cdot]$ is more superior. Sometimes it is convenient to choose $m \neq 0$.

We summarize the few steps for the calculations of the conservation laws of IDDEs. First, we have to write the IDDEs into the discrete Lax equation (23). This step will give the detailed expression of $\hat{P}$ and $\hat{L}$. Second, we get the nth root of $\hat{L}$ by (25). Third, we obtain $\hat{Y}_k$ by (27). At last, we calculate the conservation laws by the residue formula (29).

3. Applications of the theory

In this section, we present two illustrative examples. These examples are applications of the theory introduced by section 2. Moreover, we introduce an additional technique to help calculate the residues of PDOs. Some numerical results are also given to show the validity of the theory.

In the first example, we find that the residue formula is not equivalent to the trace formula. In the second example, we find that the PDOs are crucial to the construction of IDDEs.

3.1. Semi-discrete KdV equation

The Lax operator of KdV equation is $L = \partial^2 + u(x)$. But the operator $\hat{L}$ of semi-discrete KdV cannot take such a simple form. In fact, $\hat{L}$ has the form

$$\hat{L} = \Delta^2 + v\Delta + u, \tag{32}$$

where u, v are N-dimensional vectors. For convenience, we take

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{L} = [\hat{P}, \hat{L}] \tag{33}$$

as the motion equation. Suppose the square root of $\hat{L}$ is

$$\hat{L}^{\frac{1}{2}} = \Delta + \bar{l}_0 + \sum_{i=1}^{\infty} \bar{l}_i \Delta^{-i}. \tag{34}$$

According to (25) ($n = 2, m = 0$), we have

$$\hat{L}^{\frac{1}{2}}\hat{L}^{\frac{1}{2}} = \hat{L}. \tag{35}$$

By comparing the two sides of (35), we get

$$\begin{aligned}
&\bar{l}_0 + (\bar{l}_0)_+ = v,\\
&\frac{(\bar{l}_0)_+ - \bar{l}_0}{h} + (\bar{l}_0)^2 + \bar{l}_1 + (\bar{l}_1)_+ = u,\\
&\frac{(\bar{l}_1)_+ - \bar{l}_1}{h} + (\bar{l}_0 + (\bar{l}_0)_-)\bar{l}_1 + \bar{l}_2 + (\bar{l}_2)_+ = 0,\\
&\cdots + \bar{l}_3 + (\bar{l}_3)_+ = 0,\\
&\cdots,
\end{aligned} \tag{36}$$

where $(\bar{l}_0)_+ = \hat{R}\cdot\bar{l}_0$, $(\bar{l}_0)_- = \hat{R}^{-1}\cdot\bar{l}_0$, $(\bar{l}_2)_+ = \hat{R}\cdot\bar{l}_2$ and so on. Note that for any vector v, $(v)_{n+}(i) = v(n+i)$, where $v(i)$ denotes the ith component of v.

By (30), the operator $\hat{P}$ is determined by

$$\begin{aligned}
\hat{P} = {}&\Delta^3 + (v + (\bar{l}_0)_{2+})\Delta^2 + \left(u + v(\bar{l}_0)_+ + (\bar{l}_1)_{2+} + 2\frac{(\bar{l}_0)_{2+} - (\bar{l}_0)_+}{h}\right)\Delta\\
&+ \bar{l}_2 + u\bar{l}_0 + v(\bar{l}_1)_+ + v\frac{(\bar{l}_0)_+ - \bar{l}_0}{h} + 2\frac{(\bar{l}_1)_{2+} - (\bar{l}_1)_+}{h} + \frac{\bar{l}_0 - 2(\bar{l}_0)_+ + (\bar{l}_0)_{2+}}{h^2},
\end{aligned} \tag{37}$$

where $(\bar{l}_0)_{2+} = \hat{R}^2\cdot\bar{l}_0$, $(\bar{l}_1)_{2+} = \hat{R}^2\cdot\bar{l}_1$. At last by substituting (37) and (32) into (33), we obtain the differential equations

$$\begin{aligned}
\dot{v} = {}&(\bar{l}_0)_+^2(\bar{l}_1)_{2+} + (\bar{l}_1)_+(\bar{l}_1)_{2+} + (\bar{l}_1)_{2+}^2 - 2\bar{l}_0(\bar{l}_2)_+ - \bar{l}_0^2\bar{l}_1 - \bar{l}_1^2 - \bar{l}_0^2(\bar{l}_1)_+ - 2\bar{l}_0(\bar{l}_0)_+(\bar{l}_1)_+\\
&- (\bar{l}_0)_+^2(\bar{l}_1)_+ - \bar{l}_1(\bar{l}_1)_+ - 2(\bar{l}_0)_+(\bar{l}_2)_+ + \frac{2}{h}(\bar{l}_0\bar{l}_1 + \bar{l}_0(\bar{l}_1)_+ + (\bar{l}_0)_+(\bar{l}_1)_+\\
&- (\bar{l}_0)_+(\bar{l}_1)_{2+} + 2(\bar{l}_2)_+) + \frac{(\bar{l}_1)_{2+} - \bar{l}_1}{h^2}\\
\dot{u} = {}&\cdots = \frac{\dot{v}}{h}
\end{aligned} \tag{38}$$

for the semi-discrete KdV system.

Suppose the number of grid points N is odd and let M be an $N \times N$ matrix

$$M = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0\\ 0 & 1 & 1 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & 1 & 1\\ 1 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

By solving the first equation of (36), we immediately get

$$\bar{l}_0 = M^{-1}v. \tag{39}$$

By substituting (39) into the second equation of (36) we immediately obtain $\bar{l}_1$. By substituting l_0 and l_1 into the third equation of (36), we obtain $\bar{l}_2$. By substituting l_0, l_1 and l_2 into the fourth equation of (36), we obtain $\bar{l}_3$, and so on and so forth. Therefore, we have explicit expressions of $\bar{l}_k$ $(k = 0, 1, 2, \ldots)$ as a function of u and v. So, the semi-discrete KdV system (38) is the differential equations of u and v. Of course, we can as well regard (38) as the differential equations of $\bar{l}_0$ and $\bar{l}_1$. From the second equation of (38), we know v must not be zero, else u is constant and the system is trivial. If N is even, M will not have an inverse. So the semi-discrete KdV system (38) must be different from the usual discretization of KdV equation where N is arbitrary.

According to the residue formula (29), the conservation laws can be calculated by

$$j_{\frac{s}{2}} = \sum_{i=0}^{N-1} \hat{R}^i \cdot \mathrm{Res}(\hat{L}^{\frac{s}{2}} \hat{R}^{-1}), \qquad s \in Z_+. \tag{40}$$

It is thoughtless to expand $\hat{R}^{-1}$ in (40) as the power series of Δ. We need an additional technique to calculate the residue of $\hat{L}^{\frac{s}{2}} \hat{R}^{-1}$. Let $\hat{A} = \sum_{i=-\infty}^{n} a_i \Delta^i$. The residue of $\hat{A}\hat{R}^{-1}$ can be determined by the following proposition.

Proposition 1.

$$\mathrm{Res}(\hat{A}\hat{R}^{-1}) = \frac{1}{h} \sum_{i=0}^{n} \left(\frac{-1}{h}\right)^i a_i. \tag{41}$$

Proof. Let $\hat{B} = \hat{A}\hat{R}^{-1}$. Then, we have

$$\hat{A} = \hat{B}\hat{R} = \hat{B}(1 + h\Delta). \tag{42}$$

So, we can assume that $\hat{B} = \sum_{i=-\infty}^{n-1} b_i \Delta^i$. Substituting the expansion of $\hat{A}$ and $\hat{B}$ into (42), comparing the expansion of both sides of (42), we obtain $b_{-1} = \frac{1}{h} \sum_{i=0}^{n} \left(\frac{-1}{h}\right)^i a_i$. That is just (41). □

Now we can calculate the conservation laws by (40). The first few conserved quantities are

$$\begin{aligned}
j_{\frac{1}{2}} &= \sum_{i=0}^{N-1} \bar{l}_0(i),\\
j_{\frac{2}{2}} &= \sum_{i=0}^{N-1} u(i) - v(i)/h,\\
j_{\frac{3}{2}} &= \sum_{i=0}^{N-1} (\bar{l}_0(i) + v(i))/h^2 - (\bar{l}_1(i) + u_i + \bar{l}_0(i)v(i))/h + (\bar{l}_2(i) + \bar{l}_0(i)u(i) + v(i-1)\bar{l}_1(i)),\\
j_{\frac{4}{2}} &= \sum_{i=0}^{N-1} u(i)^2 - 2u(i)v(i)/h + (2u(i) + v(i)^2)/h^2 - 2v(i)/h^3.
\end{aligned} \tag{43}$$

Note that $\bar{l}_0(i)$ is the ith component of vector $\bar{l}_0$ and so on. We take the third equation of (43) as an example to show the calculation details. We know

$$\hat{L}^{\frac{3}{2}} = \hat{L}^{\frac{1}{2}} \hat{L}. \tag{44}$$

By substituting (34) and (32) into (44), we obtain

$$\hat{L}^{\frac{3}{2}} = \Delta^3 + (\bar{l}_0 + \hat{R}\cdot v)\Delta^2 + (\Delta\cdot v + \hat{R}\cdot u + \bar{l}_0 v + \bar{l}_1)\Delta + (\Delta\cdot u + \bar{l}_0 u + \bar{l}_1 \hat{R}^{-1}\cdot v + \bar{l}_2) + \cdots. \tag{45}$$

According to proposition 1, we know

$$\mathrm{Res}(\hat{L}^{\frac{3}{2}}\hat{R}^{-1}) = \frac{1}{h}\Big((\Delta\cdot u + \bar{l}_0 u + \bar{l}_1\hat{R}^{-1}\cdot v + \bar{l}_2) - \frac{1}{h}(\Delta\cdot v + \hat{R}\cdot u + \bar{l}_0 v + \bar{l}_1) + \frac{1}{h^2}(\bar{l}_0 + \hat{R}\cdot v) - \frac{1}{h^3}\Big). \tag{46}$$

By (40), we know

$$j_{\frac{3}{2}} = \sum_{i=0}^{N-1}\hat{R}^i\cdot\mathrm{Res}\big(\hat{L}^{\frac{3}{2}}\hat{R}^{-1}\big). \tag{47}$$

Then by substituting (46) into (47), we obtain

$$j_{\frac{3}{2}} = \frac{1}{h}\sum_{i=0}^{N-1}\Big(((\Delta\cdot u)(i) + \bar{l}_0(i)u(i) + \bar{l}_1(i)v(i-1) + \bar{l}_2(i)) - \frac{1}{h}((\Delta\cdot v)(i) + u(i+1) + \bar{l}_0(i)v(i) + \bar{l}_1(i)) + \frac{1}{h^2}(\bar{l}_0(i) + v(i+1)) - \frac{1}{h^3}\Big). \tag{48}$$

Note $(\Delta\cdot v)(i) = (v(i+1) - v(i))/h$. So $\sum_{i=0}^{N-1}(\Delta\cdot v)(i) = 0$. For the same reason, the sum $\sum_{i=0}^{N-1}(\Delta\cdot u)(i)$ is also zero. Therefore, (48) becomes

$$j_{\frac{3}{2}} = \frac{1}{h}\sum_{i=0}^{N-1}\Big((\bar{l}_0(i)u(i) + \bar{l}_1(i)v(i-1) + \bar{l}_2(i)) - \frac{1}{h}(u(i+1) + \bar{l}_0(i)v(i) + \bar{l}_1(i)) + \frac{1}{h^2}(\bar{l}_0(i) + v(i+1)) - \frac{1}{h^3}\Big). \tag{49}$$

If we redefine $j_{\frac{3}{2}}$ as

$$j_{\frac{3}{2}} = \sum_{i=0}^{N-1}\Big((\bar{l}_0(i)u(i) + \bar{l}_1(i)v(i-1) + \bar{l}_2(i)) - \frac{1}{h}(u(i+1) + \bar{l}_0(i)v(i) + \bar{l}_1(i)) + \frac{1}{h^2}(\bar{l}_0(i) + v(i+1))\Big), \tag{50}$$

then the new $j_{\frac{3}{2}}$ is also a conserved quantity. Equation (50) is just the third of (43).

For any positive integer s, formula (40) will provide a conserved quantity. So, formula (40) provides infinity of conserved quantities. But here the number of grid points is N, hence the number of unknown functions $\{u(i)\}\cup\{v(i)\}$ is $2N$. (Note that $u(i)$ and $v(i)$ are unknown functions of t.) In addition, variable t does not manifestly appear in formula (40). So, the number of functionally independent conserved quantities must be less than $2N$. Then, the natural question is that for given N how many functionally independent conserved quantities formula (40) provides. To see the question more clearly, let us suppose $N = 3$. Now we can expand the summation in (43)

$$j_{\frac{1}{2}} = \bar{l}_0(0) + \bar{l}_0(1) + \bar{l}_0(2),$$

$$j_{\frac{2}{2}} = u(0) + u(1) + u(2) - \frac{1}{h}(v(0) + v(1) + v(2)),$$

$$
\begin{aligned}
j_{\frac{3}{2}} = {} & \frac{1}{h^2}(v(0)+v(1)+v(2)+\bar{l}_0(0)+\bar{l}_0(1)+\bar{l}_0(2)) - \frac{1}{h}(u(0)+u(1)+u(2)+\bar{l}_1(0) \\
& +\bar{l}_1(1)+\bar{l}(2)+v(0)\bar{l}_0(0)+v(1)\bar{l}_0(1)+v(2)\bar{l}_0(2))+\bar{l}_2(0)+\bar{l}_2(1)+\bar{l}_2(2) \\
& +u(0)\bar{l}_0(0)+u(1)\bar{l}_0(1)+u(2)\bar{l}_0(2)+v(0)\bar{l}_1(1)+v(1)\bar{l}_1(2)+v(2)\bar{l}_1(0), \\
j_{\frac{4}{2}} = {} & \frac{2}{h^2}(u(0)+u(1)+u(2)) + \left(u(0)-\frac{1}{h}v(0)\right)^2 + \left(u(1)-\frac{1}{h}v(1)\right)^2 + \left(u(2)-\frac{1}{h}v(2)\right)^2 \\
& - \frac{2}{h^3}(v(0)+v(1)+v(2)). \qquad (51)
\end{aligned}
$$

Thanks to (36), we can eliminate $u(i)$, $v(i)$ and $\bar{l}_2(i)$ $(i = 0, 1, 2)$ in (51). Then, we get

$$
\begin{aligned}
& j_{\frac{1}{2}} = \bar{l}_0(0)+\bar{l}_0(1)+\bar{l}_0(2), \\
& j_{\frac{2}{2}} + \frac{2}{h}j_{\frac{1}{2}} = \bar{l}_0(0)^2+\bar{l}_0(1)^2+\bar{l}_0(2)^2+2(\bar{l}_1(0)+\bar{l}_1(1)+\bar{l}_1(2)), \\
& hj_{\frac{3}{2}} + 3j_{\frac{2}{2}} + \frac{3}{h}j_{\frac{1}{2}} = 3(\bar{l}_1(0)+\bar{l}_1(1)+\bar{l}_1(2)) + h(\cdots), \\
& hj_{\frac{4}{2}} + \frac{8}{3}j_{\frac{3}{2}} + \frac{2}{h}j_{\frac{2}{2}} = -\frac{4}{3}(\bar{l}_0(0)^3+\bar{l}_0(1)^3+\bar{l}_0(2)^3) + h(\cdots).
\end{aligned}
$$

Now it is clear that the four js are functionally independent, because the set of functions $\{\bar{l}_0(0), \bar{l}_0(1), \bar{l}_0(2), \bar{l}_1(0), \bar{l}_1(1), \bar{l}_1(2)\}$ are functionally independent. Hence, we have proved that the residue formula provides at least four functionally independent conserved quantities for the semi-discrete KdV equation with the number of grid points $N = 3$.

We emphasize that the residue formula is not equivalent to the trace formula. The trace formula is the familiar way to calculate the conserved quantities for equations in the form of (33). To apply the trace formula, equation (33) is explained as a matrix equation where $\hat{P}$ and $\hat{L}$ are both $N \times N$ matrices. Their conserved quantities are $C_k = \mathrm{Trace}(\hat{L}^k)$, $k \in N$. If $N = 3$, among $\{C_k | k \in N\}$ there are at most three functionally independent conserved quantities corresponding to the three eigenvalues of $\hat{L}$. Therefore, the trace formula can provide at most three functionally independent conserved quantities for the $N = 3$ case of semi-discrete KdV equation. Whereas we have proved that the residue formula provides at least four independent conserved quantities for the $N = 3$ case of semi-discrete KdV equation. Thus, the residue formula is not equivalent to the trace formula. The residue formula is, in our example, superior to the trace formula.

Conserved quantity $j_{\frac{5}{2}}$ is much more complicated

$$
\begin{aligned}
j_{\frac{5}{2}} = \sum_{i=0}^{N-1} \Bigg(& u(i)^2\bar{l}_0(i) + u(i)v(i)\bar{l}_1(i+1) + u(i+1)v(i)\bar{l}_1(i+1) + v(i)v(i+1)\bar{l}_2(i+2) \\
& + u(i)\bar{l}_2(i) + u(i)\bar{l}_2(i+2) + v(i)\bar{l}_3(i+3) + v(i)\bar{l}_3(i+1) + \bar{l}_4(i) \\
& + \frac{1}{h}(\cdots) + \frac{1}{h^2}(\cdots) + \frac{1}{h^3}(\cdots) + \frac{1}{h^4}(2v(i)+\bar{l}_0(i)) \Bigg). \qquad (52)
\end{aligned}
$$

We believe (52) induces the fifth functionally independent conserved quantity for $N = 3$. According to the general theory of ordinary differential equations [2], there are only $n - 1$ time-independent first integrals for n first-order ordinary differential equations. In the $N = 3$ case of semi-discrete KdV equation there are six unknown functions, so there are only five time-independent constants of motion. Hence, it will be a surprising result if the residue formula could provide all the five time-independent first integrals.

When N is small enough (such as $N = 3$), we can directly verify that js are constants of motion by verifying $\frac{\mathrm{d}}{\mathrm{d}t}j_{\frac{5}{2}} = 0$. But the expressions involved are too lengthy, we omit them

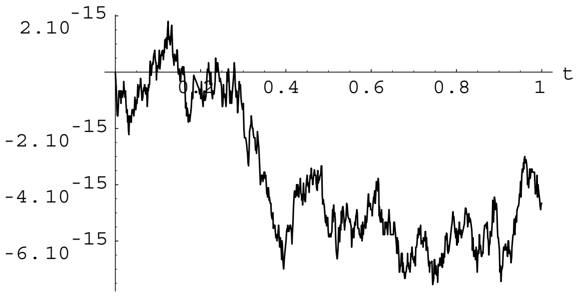

Figure 1. Constant of motion $\bar{j}_{\frac{1}{2}}$ versus time t.

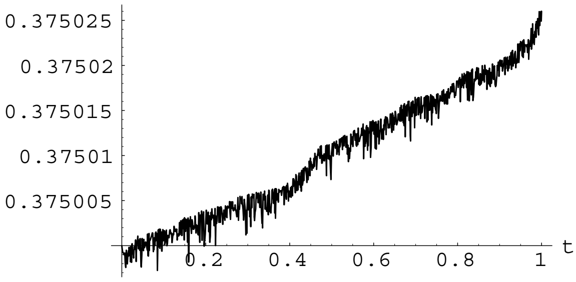

Figure 2. Constant of motion $\bar{j}_{\frac{3}{2}} - 9$ versus time t.

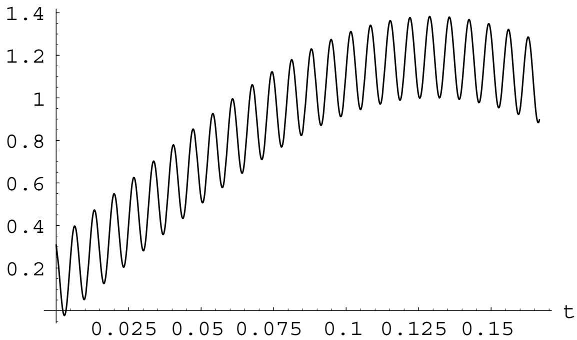

Figure 3. Evolution of $\bar{l}_0(1)$.

here. Fortunately, numerical experiments could provide a concise and convincing verification of the theory (even for relatively large N).

Figures 1–3 show the numerical results of the semi-discrete KdV (38) with parameters $N = 5, h = \frac{2}{N}$. According to (38), $u = \frac{v}{h} + V_c$, where V_c is a constant vector. In the numerical experiments we assume that $V_c = 0$. Hence, $j_{\frac{2}{2}}$ and $j_{\frac{4}{2}}$ are always zero. So, we only need to compute $j_{\frac{1}{2}}$ and $j_{\frac{3}{2}}$. Here, we computed $\bar{j}_{\frac{1}{2}} = \sum_{i=1}^{N} \bar{l}_0(i)$ and $\bar{j}_{\frac{3}{2}} = \sum_{i=1}^{N} \bar{l}_2(i) + v(i)\bar{l}_1(i) - \frac{\bar{l}_1(i)}{h}$, which are simplified versions of $j_{\frac{1}{2}}$ and $j_{\frac{3}{2}}$. The initial condition is $\bar{l}_0(i) = \cos\left(\frac{2\pi i}{N}\right)$. (Here, $\bar{l}_1$ is dependent on $\bar{l}_0$ because of the new restriction $u = \frac{v}{h}$.) The numerical results are obtained by Mathematica NDSolve. Figures 1 and 2 show that $\bar{j}_{\frac{1}{2}}$ and $\bar{j}_{\frac{3}{2}}$ are indeed constants of motion. Figure 3 indicates that the evolution time is long enough. (The results also verified the capability of the 'NDSolve' function of Mathematica.)

3.2. *An example of* $m = 1$

In this example, we take

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{L} = [\hat{P}, \hat{L}]_1 = \hat{P}\hat{L} - \hat{L}\hat{P}_+ \tag{53}$$

as motion equation. At the same time we assume that $\hat{P}$ and $\hat{L}$ have the form

$$\begin{aligned} \hat{L} &= l_0 + l_1\Delta^{-1}, \\ \hat{P} &= \Delta(p_1\Delta + p_0), \end{aligned} \tag{54}$$

where l_0, l_1, p_0 and p_1 are N-dimensional vectors. By substituting (54) into (53), we obtain p_0 and p_1

$$\begin{aligned} p_1 &= \alpha l_0 (l_0)_-, \\ p_0 &= \beta(l_0)_- + \alpha l_0\left(\frac{l_0 + (l_0)_{2-}}{h} + l_1 + (l_1)_-\right), \end{aligned} \tag{55}$$

where α and β are arbitrary complex constants.

So, the motion equation (53) becomes

$$\begin{aligned} \dot{l}_0 = \beta l_0 &\left(\frac{l_0 - (l_0)_-}{h} + (l_1)_+ - l_1\right) + \frac{\alpha}{h^2} l_0(l_0)_-(l_0 - (l_0)_{2-}) + \alpha l_0\big((l_1)_+^2 - l_1^2\big) \\ &+ \frac{\alpha}{h} l_0((l_0)_+(l_1)_{2+} + l_0(l_1)_+ + l_0 l_1 - (l_0)_-(l_1)_- - 2(l_0)_- l_1), \\ \dot{l}_1 = \frac{\beta}{h}&(l_0(l_1)_+ - (l_0)_- l_1) + \frac{\alpha}{h^2}(l_0(l_0)_+(l_1)_{2+} - (l_0)_{2-}(l_0)_- l_1) \\ &+ \frac{\alpha}{h}\big(l_0(l_1)_+^2 + l_0(l_1)_+ l_1 - (l_0)_- l_1^2 - (l_0)_- l_1(l_1)_-\big). \end{aligned} \tag{56}$$

By setting $l_0 \to v(x,t), l_1 \to \rho(x,t)$ and $\alpha \to \theta h$, we can get the continuous limit of (56):

$$\begin{aligned} v_t &= \beta v(v_x + \rho_x) + \theta(2v^2 v_x + (\rho^2)_x v + 4v(\rho v)_x), \\ \rho_t &= \beta(\rho v)_t + 2\theta((\rho v^2)_x + (\rho^2 v)_x). \end{aligned} \tag{57}$$

If we interpret ρ and v as the density and the speed of the vehicles on a road, then the $\beta = 1, \theta = 0$ case of (57) can be interpreted as a traffic toy model. The second of (57) is the continuity equation, while the first of (57) is the control strategy of the driver.

From the residue formula, we can calculate the conservation laws of (56) directly. The first few conservation laws are

$$\begin{aligned} j_1 &= \sum_{i=1}^{N} l_1(i), \\ j_2 &= \sum_{i=1}^{N} 2l_0(i)l_1(i+1) + hl_1(i)^2, \\ j_3 &= \sum_{i=0}^{N} (3l_0(i)l_0(i+1)l_1(i+2) + 3hl_0(i)l_1(i)l_1(i+1) + 3hl_0(i-1)l_1(i)^2 + h^2 l_1(i)^3), \end{aligned} \tag{58}$$

where $l_0(i)$ and $l_1(i)$ are the ith components of vector l_0 and l_1, respectively. If the grid number N is 2, we can easily verify (by setting $h \to 0$) that three js are independent. In this case, there

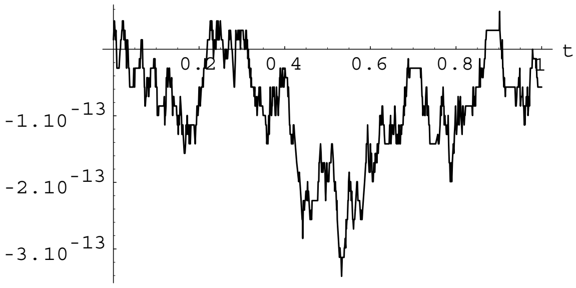

Figure 4. Constant of motion $j_1 - 120$ versus time t.

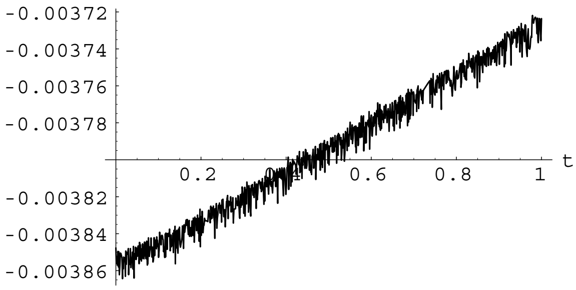

Figure 5. Constant of motion $j_2 - 1286.2$ versus time t.

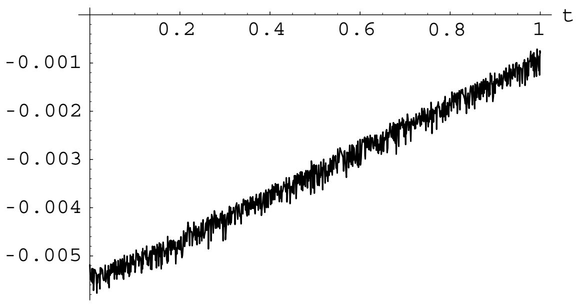

Figure 6. Constant of motion $j_3 - 16679.4$ versus t.

are four unknown functions, so there are only three time-independent first integrals. Again the theory provides all the time-independent constants of motion. If $N = 3$, the residue formula provides four time-independent first integrals. So, it seems that the residue formula provides $N + 1$ constants of motion of (56).

Equations (56), as far as we known, are new. If we artificially write (56) into

$$\frac{\mathrm{d}}{\mathrm{d}t}(l_0\Delta + l_1) = \Delta(p_1\Delta + p_0)(l_0\Delta + l_1) - (l_0\Delta + l_1)((p_1)_+\Delta + (p_0)_+)\Delta, \tag{59}$$

then we could not connect (59) with any matrix eigenvalue problems, though (59) itself can be thought as a matrix differential equation. Without PDOs we would not be able to construct (56). Therefore, the PDOs are crucial in the construction of IDDEs.

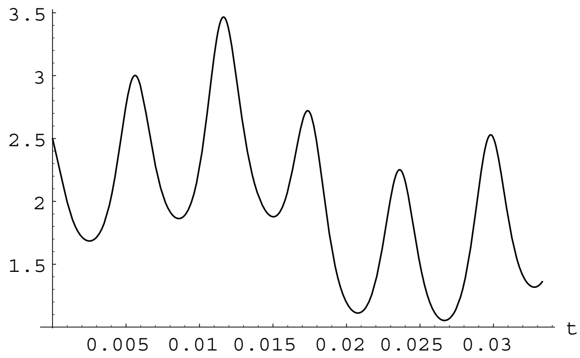

Figure 7. Evolution of $l_0(1)$.

The numerical results ($N = 6, \alpha = 1, \beta = 0, h = \frac{2}{N}$) of j_1, j_2 and j_3 are displayed in figures 4–6. Figure 7 is the graph of $l_0(1)$, which indicates that the evolution time is long enough. The initial condition is $l_0(i) = 2 + \cos\left(\frac{2\pi \mathrm{i}}{N}\right)$, $l_1(i) = 20 + \sin\left(\frac{2\pi \mathrm{i}}{N}\right)$. It is easy to see that if it were not for the computing errors, j_1, j_2 and j_3 will all be true constants. The numerical results are obtained by Mathematica NDSolve.

4. Conclusions

We have proposed a series of commutators for the construction of discrete Lax equation by the PDOs. The two most important commutators in the series are $[\cdot, \cdot]$ and $[\cdot, \cdot]_1$. We have presented a residue formula for the calculations of the conservation laws of the discrete Lax equation. We have also demonstrated the theory by two examples. The periodic boundary condition has been dealt with successfully.

We find that the residue formula is excellent in the calculations of the conservation laws of the discrete Lax system. The residue formula is not equivalent to the trace formula.

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